ALL SAINTS' COLLEGE

WA Exams Practice Paper C, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	54	35
Section Two: Calculator-assumed	13	13	100	96	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **35% (54 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

(a) Determine $\int 8\sin^2(2\theta)\ d\theta$. (3 marks)

$$\int 8\sin^2(2\theta)\ d\theta = \int 8\times\frac{1}{2}\times(1-\cos 4\theta)\ d\theta$$
$$= 4\int 1-\cos 4\theta\ d\theta$$
$$= 4\left[\theta - \frac{\sin 4\theta}{4}\right] + c$$
$$= 4\theta - \sin 4\theta + c$$

(b) Evaluate $\int_0^1 4x(x^2+1)^3 dx$ using a suitable substitution. (4 marks)

$$u = x^2+1 \ \Rightarrow\ du = 2xdx$$
$$x = 0, u = 1;\ x = 1, u = 2$$
$$\int_0^1 2(x^2+1)^3\times 2x\ dx = \int_1^2 2u^3 du$$
$$= \left[\frac{u^4}{2}\right]_1^2 = 8 - \frac{1}{2} = \frac{15}{2}$$

Question 2 **(6 marks)**

(a) If $y = \sin(x+y)$, determine an expression for $\frac{dy}{dx}$. (3 marks)

$$\frac{dy}{dx} = \left(1 + \frac{dy}{dx}\right)\cos(x+y)$$

$$\frac{dy}{dx} = \cos(x+y) + \frac{dy}{dx}\cos(x+y)$$

$$\frac{dy}{dx}\left(1 - \cos(x+y)\right) = \cos(x+y)$$

$$\frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)}$$

(b) Determine the gradient of the curve $y = xy + \frac{1}{x}$ at the point where $x = 2$. (3 marks)

When $x = 2,\ y = 2y + \frac{1}{2} \ \Rightarrow\ y = -\frac{1}{2}$

$$\frac{dy}{dx} = y + x\frac{dy}{dx} - \frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{2} + 2\frac{dy}{dx} - \frac{1}{2^2}$$

$$\frac{dy}{dx} = \frac{3}{4}$$

Question 3 **(7 marks)**

(a) Given that $w=1+i$, determine w^2 and hence calculate w^8. (2 marks)

$$w^2 = (1+i)(1+i) = 1+2i+i^2 = 2i$$

$$w^8 = \left(w^2\right)^4 = \left(2i\right)^4 = 16i^4 = 16$$

(b) Consider $f(z)=2z^3-5z^2+22z-10$, $z\in\mathbb{C}$.

(i) Show that $f\left(0.5\right)=0$. (1 mark)

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{8}\right)-5\left(\frac{1}{4}\right)+22\left(\frac{1}{2}\right)-10 \\ &= \frac{1}{4}-\frac{5}{4}+11-10 \\ &= -1+1 \\ &= 0 \end{aligned}$$

(ii) Find all values, real and complex, for which $f(z)=0$. (4 marks)

$$\frac{2z^3-5z^2+22z-10}{z-\frac{1}{2}} = 2z^2-4z+20$$

$$z^2-2z+10=0$$
$$(z-1)^2-1+10=0$$

$$(z-1)^2=-9$$
$$z-1=\pm 3i$$

$$z=1\pm 3i$$

$$f(z)=0 \;\Rightarrow\; z=\frac{1}{2},\ 1+3i,\ 1-3i$$

Question 4 **(7 marks)**

The function $f(x)$ is defined by $f(x)=\dfrac{x^2-5x+4}{x+1}$.

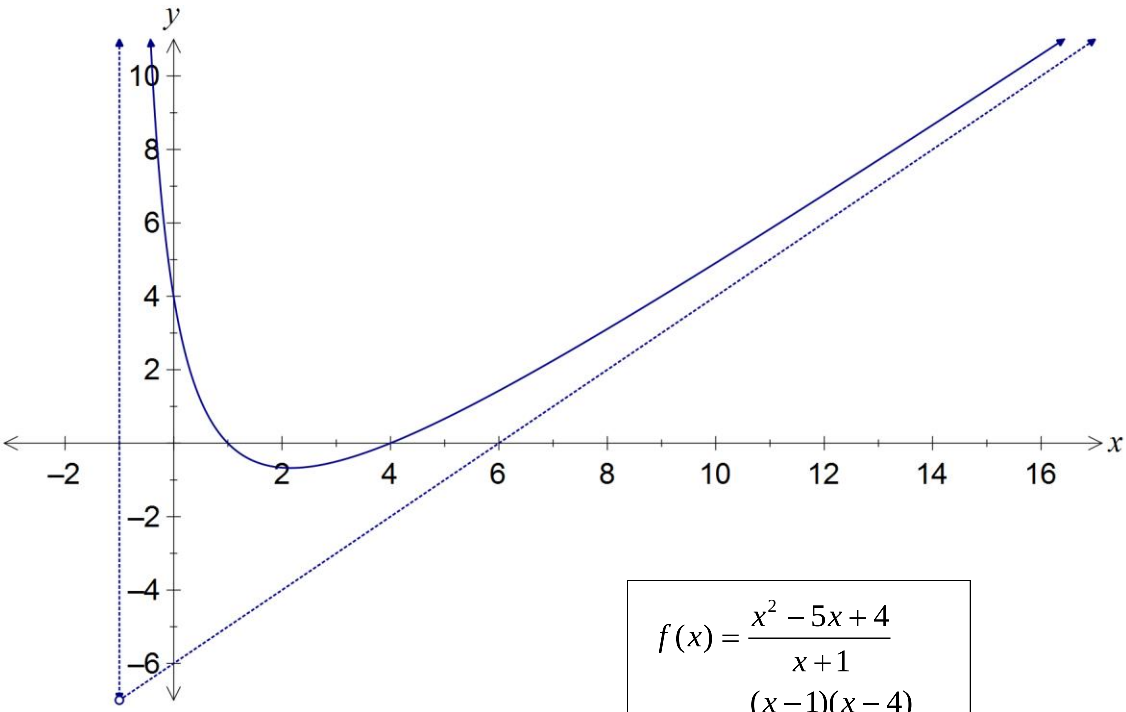
(a) Show that $y=x-6$ is an asymptote of the curve $y=f(x)$. (2 marks)

$$f(x)=\frac{x^2-5x+4}{x+1}$$
$$=x-6+\frac{10}{x+1} \Rightarrow x\to\infty,\ y\to x-6$$

(b) State the equation of the other asymptote of the graph. (1 mark)

$x=-1$

(c) Sketch the graph of $y=f(x)$, $x\geq -1$, on the axes below, clearly showing the asymptotes and the points of intersection of the curve with the coordinate axes. (4 marks)



Marks:
- roots
- asymptotes
- y-intercept
- shape

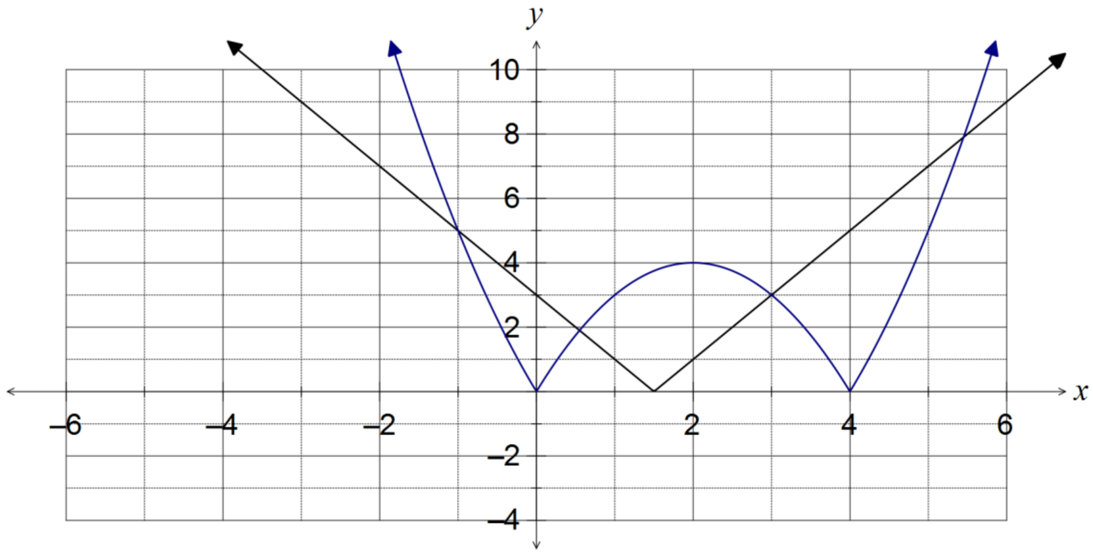
$$f(x)=\frac{x^2-5x+4}{x+1}$$
$$=\frac{(x-1)(x-4)}{x+1}$$
$$f(x)=0 \Rightarrow x=1,\ 4$$

$$f(0)=4$$

$$x\to -1^{+}, y\to +\infty$$

Question 5 **(6 marks)**

The graph of $y = |f(x)|$ is shown, where $f(x) = 2x - 3$.



(a) Add the graph of $y = |g(x)|$ to the axes above, where $g(x) = (x-2)^2 - 4$. (2 marks)

(b) Solve $|f(x)| = |g(x)|$. (4 marks)

$(x-2)^2 - 4 = x^2 - 4x$

$x^2 - 4x = -2x + 3$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0 \Rightarrow x = -1,\ 3$ (or from graph)

$x^2 - 4x = 2x - 3$

$x^2 - 6x + 3 = 0$

$(x-3)^2 = 6 \Rightarrow x = 3 \pm \sqrt{6}$

$x = -1,\ 3-\sqrt{6},\ 3,\ 3+\sqrt{6}$

Question 6 **(7 marks)**

A particle moves in a straight line so that after t seconds it has a displacement of x cm from a fixed point O. The acceleration of the particle is $3x^2$ cms^{-2} and the velocity is zero when $x = k$.

(a) Express the velocity of the particle in terms of x and k. (4 marks)

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = 3x^2$$

$$\int v\,dv = \int 3x^2\,dx$$

$$\frac{v^2}{2} = x^3 + c_1$$

$$v^2 = 2x^3 + c_2$$

$$0 = 2k^3 + c_2$$

$$v^2 = 2x^3 - 2k^3$$

$$v = \pm\sqrt{2x^3 - 2k^3},\ x \ge k$$

(b) If $k = -1$, determine the change in displacement of the particle as the velocity increases from rest to 0.5 ms^{-1}. (3 marks)

$$v = 0,\ x = -1$$

$$v^2 = 2x^3 - 2(-1)^3$$

$$= 2x^3 + 2$$

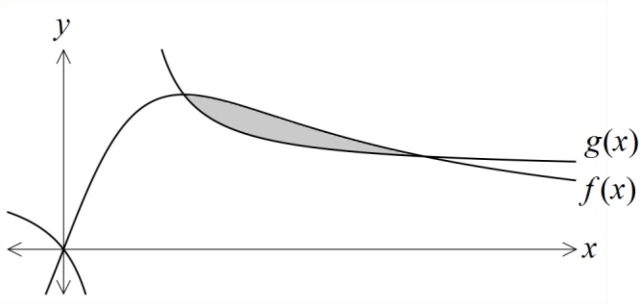
$$\left(\frac{1}{2}\right)^2 = 2x^3 + 2$$

$$x^3 = \frac{1}{8}$$

$$x = \frac{1}{2} \Rightarrow \Delta x = \frac{1}{2} - (-1) = \frac{3}{2} \text{ m}$$

Question 7 **(7 marks)**

The shaded region below is enclosed by the functions $f(x)=\dfrac{2x}{x^2+1}$ and $g(x)=\dfrac{x}{2x-1}$.



Show that the area of the shaded region is $\dfrac{3\ln 5}{4}-1$ square units.

$$\frac{2x}{x^2+1}=\frac{x}{2x-1} \Rightarrow 4x^2-2x=x^3+x$$

$$x(x^2-4x+3)=0$$
$$x(x-1)(x-3)=0$$
$$\Rightarrow x=0,\ 1,\ 3$$

$$\frac{x}{2x-1}=\frac{A}{2x-1}+\frac{B}{1}$$
$$x=A+2Bx-B \Rightarrow B=\tfrac{1}{2}, A=\tfrac{1}{2}$$

$$A=\int_1^3 \frac{2x}{x^2+1}-\left(\frac{\frac{1}{2}}{2x-1}+\frac{1}{2}\right)\,dx$$

$$=\left[\ln\left(x^2+1\right)-\frac{1}{4}\ln|2x-1|-\frac{x}{2}\right]_1^3$$

$$=\left(\ln 10-\frac{1}{4}\ln 5-\frac{3}{2}\right)-\left(\ln 2-\frac{1}{4}\ln 1-\frac{1}{2}\right)$$

$$=\ln 10-\ln 2-\frac{1}{4}\ln 5-1$$
$$=\ln 5-\frac{1}{4}\ln 5-1$$
$$=\frac{3}{4}\ln 5-1$$

Question 8 **(7 marks)**

The two variables x and y, where $-2 < x < 4$ and $y > 0$, are related by the differential equation $y - x\dfrac{dy}{dx} = 2\dfrac{dy}{dx} - 2y^2$, subject to the initial condition that $y = \frac{1}{2}$ when $x = 1$.

Solve the differential equation for y and hence determine the value of y when $x = 2$.

$$y - x\frac{dy}{dx} = 2\frac{dy}{dx} - 2y^2$$

$$2y^2 + y = (2 + x)\frac{dy}{dx}$$

$$\int \frac{dx}{x+2} = \int \frac{dy}{y(1+2y)}$$

$$\int \frac{1}{x+2}\ dx = \int \frac{1}{y} - \frac{2}{1+2y}\ dy$$

$$\ln(x+2) = \ln(y) - \ln(1+2y) + \ln c$$

$$x + 2 = \frac{cy}{1+2y}$$

$$x = 1, y = \tfrac{1}{2} \ \Rightarrow\ 3 = \frac{c}{4} \ \Rightarrow\ c = 12$$

$$x + 2 = \frac{12y}{1+2y}$$

$$x + 2 + 2xy + 4y = 12y \ \Rightarrow\ y = \frac{x+2}{8-2x}$$

$$x = 2,\ y = \frac{4}{4} = 1$$

End of questions

Additional working space

Question number: _________